

# Probabilistic entailment in the setting of coherence: The role of quasi conjunction and inclusion relation <sup>☆</sup>

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## Abstract

In this paper, by adopting a coherence-based probabilistic approach to default reasoning, we focus the study on the logical operation of quasi conjunction and the Goodman-Nguyen inclusion relation for conditional events. We recall that quasi conjunction is a basic notion for defining consistency of conditional knowledge bases. By deepening some results given in a previous paper we show that, given any finite family of conditional events  $\mathcal{F}$  and any nonempty subset  $\mathcal{S}$  of  $\mathcal{F}$ , the family  $\mathcal{F}$  p-entails the quasi conjunction  $\mathcal{C}(\mathcal{S})$ ; then, given any conditional event  $E|H$ , we analyze the equivalence between p-entailment of  $E|H$  from  $\mathcal{F}$  and p-entailment of  $E|H$  from  $\mathcal{C}(\mathcal{S})$ , where  $\mathcal{S}$  is some nonempty subset of  $\mathcal{F}$ . We also illustrate some alternative theorems related with p-consistency and p-entailment. Finally, we deepen the study of the connections between the notions of p-entailment and inclusion relation by introducing for a pair  $(\mathcal{F}, E|H)$  the (possibly empty) class  $\mathcal{K}$  of the subsets  $\mathcal{S}$  of  $\mathcal{F}$  such that  $\mathcal{C}(\mathcal{S})$  implies  $E|H$ . We show that the class  $\mathcal{K}$  satisfies many properties; in particular  $\mathcal{K}$  is additive and has a greatest element which can be determined by applying a suitable algorithm.

*Keywords:* Coherence, Probabilistic default reasoning, p-entailment, quasi conjunction, Goodman-Nguyen's inclusion relation, QAND rule

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## 1. Introduction

Probabilistic reasoning is a basic tool for the treatment of uncertainty in many applications of statistics and artificial intelligence; in particular, it is useful for a flexible numerical approach to inference rules in nonmonotonic reasoning, for the psychology of uncertain reasoning and for the management of uncertainty on semantic web (see, e.g., [21, 29, 35, 39, 40, 42, 43]).

This work concerns nonmonotonic reasoning, an important topic in the field of artificial intelligence which has been studied by many authors, by using symbolic and/or numerical formalisms (see, e.g. [3, 5, 6, 16, 19, 28, 37]). As is well known, differently from classical (monotonic) logic, in (nonmonotonic) commonsense reasoning if a conclusion  $C$  follows from some premises, then  $C$  may be retracted when the set of premises is enlarged; that is, adding premises may invalidate previous conclusions. Among the numerical formalisms connected with nonmonotonic reasoning, a remarkable theory is represented by the Adams probabilistic logic of conditionals ([1]), which can be developed with full generality in the setting of coherence. As is well known, based on the coherence principle of de Finetti ([20]), conditional probabilities can be directly assigned to conditional assertions, without assuming that conditioning events have a positive probability (see, e.g., [8, 9, 10, 15, 16, 26, 44]). We also recall that this approach does not require the assertion of complete distributions and is largely applied in statistical analysis and decision theory (see, for instance, [2, 11, 12, 13, 14, 17, 38]). In Adams' work a basic notion is the quasi conjunction of conditionals, which has a strict relationship with the property of consistency of conditional knowledge bases. Quasi conjunction also plays a relevant role in the work of Dubois and Prade on conditional objects ([19], see also [3]), where a suitable QAND rule has been introduced to characterize entailment from a conditional knowledge base. Recently ([30], see also [34]), we have studied some probabilistic aspects related with the QAND rule and with the conditional probabilistic logic of Adams. We continue such a study in this paper by giving further results on the role played by quasi conjunction and Goodman-Nguyen's inclusion relation in the probabilistic entailment under coherence.

The paper is organized as follows: In Section 2 we first recall some notions and results on coherence; then, we recall basic notions in probabilistic default reasoning; we recall the operation of quasi conjunction and the inclusion relation for conditional events; finally, we recall the notion of entailment for conditional objects. In Section 3 we give a result which shows the p-entailment

from a family of conditional events  $\mathcal{F}$  to the quasi conjunction  $\mathcal{C}(\mathcal{S})$ , for every nonempty subset  $\mathcal{S}$  of  $\mathcal{F}$ ; we give another result which analyzes many aspects connected with the equivalence between the p-entailment of a conditional event  $E|H$  from  $\mathcal{F}$  and the p-entailment of  $E|H$  from  $\mathcal{C}(\mathcal{S})$ , where  $\mathcal{S}$  is some nonempty subset of  $\mathcal{F}$ ; then, we give some alternative theorems related with p-consistency and p-entailment. In Section 4 we introduce for a pair  $(\mathcal{F}, E|H)$  the class  $\mathcal{K}$  of the subsets  $\mathcal{S}$  of  $\mathcal{F}$  such that  $\mathcal{C}(\mathcal{S})$  implies  $E|H$ . We show that  $\mathcal{K}$  satisfies many properties and we give some examples; in particular, we show that  $\mathcal{K}$  is additive and has a greatest element (if any) which can be determined by a suitable algorithm. In Section 5 we give some conclusions.

## 2. Some Preliminary Notions

In this section we recall some basic notions and results on the following topics: (i) coherence of conditional probability assessments; (ii) probabilistic default reasoning; (iii) inclusion relation of Goodman-Nguyen and quasi conjunction of conditional events; (iv) entailment among conditional objects and QAND rule.

### 2.1. Basic notions on coherence

Given any event  $E$  we use the same symbol to denote its indicator and we denote by  $E^c$  the negation of  $E$ . Given any events  $A$  and  $B$ , we simply write  $A \subseteq B$  to denote that  $A$  logically implies  $B$ . Moreover, we denote by  $AB$  (resp.,  $A \vee B$ ) the logical intersection, or conjunction (resp., logical union, or disjunction) of  $A$  and  $B$ . We recall that  $n$  events are said logically independent when there are no logical dependencies among them; this amounts to say that the number of atoms, or constituents, generated by them is  $2^n$ . The conditional event  $B|A$ , with  $A \neq \emptyset$ , is looked at as a three-valued logical entity which is true, or false, or void, according to whether  $AB$  is true, or  $AB^c$  is true, or  $A^c$  is true. Given a real function  $P : \mathcal{F} \rightarrow \mathbb{R}$ , where  $\mathcal{F}$  is an arbitrary family of conditional events, let us consider a subfamily  $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\} \subseteq \mathcal{F}$ , and the vector  $\mathcal{P}_n = (p_1, \dots, p_n)$ , where  $p_i = P(E_i|H_i)$ ,  $i = 1, \dots, n$ . We denote by  $\mathcal{H}_n$  the disjunction  $H_1 \vee \dots \vee H_n$ . Notice that,  $E_i H_i \vee E_i^c H_i \vee H_i^c = \Omega$ ,  $i = 1, \dots, n$ , where  $\Omega$  is the sure event; then, by expanding the expression  $\bigwedge_{i=1}^n (E_i H_i \vee E_i^c H_i \vee H_i^c)$ , we can represent  $\Omega$  as the disjunction of  $3^n$  logical conjunctions, some of which may be impossible. The remaining ones are the constituents generated by the family  $\mathcal{F}_n$ .

We denote by  $C_1, \dots, C_m$  the constituents contained in  $\mathcal{H}_n$  and (if  $\mathcal{H}_n \neq \Omega$ ) by  $C_0$  the further constituent  $\mathcal{H}_n^c = H_1^c \cdots H_n^c$ , so that

$$\mathcal{H}_n = C_1 \vee \cdots \vee C_m, \quad \Omega = \mathcal{H}_n^c \vee \mathcal{H}_n = C_0 \vee C_1 \vee \cdots \vee C_m, \quad m+1 \leq 3^n.$$

With the pair  $(\mathcal{F}_n, \mathcal{P}_n)$  we associate the random gain  $\mathcal{G} = \sum_{i=1}^n s_i H_i (E_i - p_i)$ , where  $s_1, \dots, s_n$  are  $n$  arbitrary real numbers. Let  $g_h$  be the value of  $\mathcal{G}$  when  $C_h$  is true; of course  $g_0 = 0$ . Denoting by  $\mathcal{G}|\mathcal{H}_n$  the restriction of  $\mathcal{G}$  to  $\mathcal{H}_n$ , it is  $\mathcal{G}|\mathcal{H}_n \in \{g_1, \dots, g_m\}$ . Then, we have

**Definition 1.** The function  $P$  defined on  $\mathcal{F}$  is *coherent* if and only if, for every integer  $n$ , for every finite sub-family  $\mathcal{F}_n \subseteq \mathcal{F}$  and for every  $s_1, \dots, s_n$ , one has:  $\min \mathcal{G}|\mathcal{H}_n \leq 0 \leq \max \mathcal{G}|\mathcal{H}_n$ .

From the previous definition it immediately follows that in order  $P$  be coherent it must be  $P(E|H) \in [0, 1]$  for every  $E|H \in \mathcal{F}$ . If  $P$  is coherent it is called a *conditional probability on  $\mathcal{F}$*  (see, e.g., [16]). Given any family  $\mathcal{F}^*$ , with  $\mathcal{F} \subset \mathcal{F}^*$ , and any function  $P^*$  defined on  $\mathcal{F}^*$ , assuming  $P$  coherent, we say that  $P^*$  is a coherent extension of  $P$  if the following conditions are satisfied: (i)  $P^*$  is coherent; (ii) the restriction of  $P^*$  to  $\mathcal{F}$  coincides with  $P$ , that is for every  $E|H \in \mathcal{F}$  it holds that  $P^*(E|H) = P(E|H)$ . In particular, if  $\mathcal{F}^*$  contains the set of unconditional events  $\{EH, H : E|H \in \mathcal{F}\}$ , then for every  $E|H \in \mathcal{F}$  the coherent extension  $P^*$  satisfies the compound probability theorem  $P^*(EH) = P^*(H)P(E|H)$  and hence, when  $P^*(H) > 0$ , we can represent  $P(E|H)$  as the ratio  $\frac{P^*(EH)}{P^*(H)}$ .

With each  $C_h$  contained in  $\mathcal{H}_n$  we associate a point  $Q_h = (q_{h1}, \dots, q_{hn})$ , where  $q_{hj} = 1$ , or  $0$ , or  $p_j$ , according to whether  $C_h \subseteq E_j H_j$ , or  $C_h \subseteq E_j^c H_j$ , or  $C_h \subseteq H_j^c$ . Denoting by  $\mathcal{I}$  the convex hull of  $Q_1, \dots, Q_m$ , based on the penalty criterion, the result below can be proved ([22, 23], see also [31]).

**Theorem 1.** The function  $P$  is coherent if and only if, for every finite sub-family  $\mathcal{F}_n \subseteq \mathcal{F}$ , one has  $\mathcal{P}_n \in \mathcal{I}$ .

The condition  $\mathcal{P}_n \in \mathcal{I}$  amounts to solvability of the following system  $\Sigma$  in the unknowns  $\lambda_1, \dots, \lambda_m$

$$(\Sigma) \quad \begin{cases} \sum_{h=1}^m q_{hj} \lambda_h = p_j, & j = 1, \dots, n; \\ \sum_{h=1}^m \lambda_h = 1, & \lambda_h \geq 0, \quad h = 1, \dots, m. \end{cases}$$

*Checking coherence of the assessment  $\mathcal{P}_n$  on  $\mathcal{F}_n$ .*

Let  $S$  be the set of solutions  $\Lambda = (\lambda_1, \dots, \lambda_m)$  of the system  $\Sigma$ . Then, define

$$\begin{aligned}\Phi_j(\Lambda) &= \Phi_j(\lambda_1, \dots, \lambda_m) = \sum_{r: C_r \subseteq H_j} \lambda_r, \quad j = 1, \dots, n; \\ M_j &= \max_{\Lambda \in S} \Phi_j(\Lambda), \quad j = 1, \dots, n; \quad I_0 = \{j : M_j = 0\}.\end{aligned}\tag{1}$$

Notice that  $I_0 \subset \{1, \dots, n\}$ , where  $\subset$  means strict inclusion. We denote by  $(\mathcal{F}_0, \mathcal{P}_0)$  the pair associated with  $I_0$ , that is  $\mathcal{F}_0 = \{E_j | H_j \in \mathcal{F}_n : j \in I_0\}$  and  $\mathcal{P}_0 = (p_j : j \in I_0)$ . Given the pair  $(\mathcal{F}_n, \mathcal{P}_n)$  and a subset  $J \subset \{1, \dots, n\}$ , we denote by  $(\mathcal{F}_J, \mathcal{P}_J)$  the pair associated with  $J$  and by  $\Sigma_J$  the corresponding system. We observe that  $\Sigma_J$  is solvable if and only if  $\mathcal{P}_J \in \mathcal{I}_J$ , where  $\mathcal{I}_J$  is the convex hull associated with the pair  $(\mathcal{F}_J, \mathcal{P}_J)$ . Then, we have ([24, 25], see also [7])

**Theorem 2.** Given a probability assessment  $\mathcal{P}_n$  on the family  $\mathcal{F}_n$ , if the system  $\Sigma$  associated with  $(\mathcal{F}_n, \mathcal{P}_n)$  is solvable, then for every  $J \subset \{1, \dots, n\}$ , such that  $J \setminus I_0 \neq \emptyset$ , the system  $\Sigma_J$  associated with  $(\mathcal{F}_J, \mathcal{P}_J)$  is solvable too.

By the previous results, we obtain

**Theorem 3.** The assessment  $\mathcal{P}_n$  on  $\mathcal{F}_n$  is coherent if and only if the following conditions are satisfied: (i)  $\mathcal{P}_n \in \mathcal{I}$ ; (ii) if  $I_0 \neq \emptyset$ , then  $\mathcal{P}_0$  is coherent.

Then, we can check coherence by the following procedure:

**Algorithm 1.** Let the pair  $(\mathcal{F}_n, \mathcal{P}_n)$  be given.

1. Construct the system  $\Sigma$  and check its solvability.
2. If the system  $\Sigma$  is not solvable then  $\mathcal{P}_n$  is not coherent and the procedure stops, otherwise compute the set  $I_0$ .
3. If  $I_0 = \emptyset$  then  $\mathcal{P}_n$  is coherent and the procedure stops; otherwise set  $(\mathcal{F}_n, \mathcal{P}_n) = (\mathcal{F}_0, \mathcal{P}_0)$  and repeat steps 1-3.

We remark that, in the algorithm,  $\Sigma$  is initially the system associated with  $(\mathcal{F}_n, \mathcal{P}_n)$ ; after the first cycle  $\Sigma$  is the system associated with  $(\mathcal{F}_0, \mathcal{P}_0)$ , and so on. If, after  $k + 1$  cycles, the algorithm stops at step 2 because  $\Sigma$  is unsolvable, then denoting by  $(\mathcal{F}_k, \mathcal{P}_k)$  the pair associated with  $\Sigma$ , we have that  $\mathcal{P}_k$  is not coherent and, of course,  $\mathcal{P}_n$  is not coherent too.

## 2.2. Basic notions on probabilistic default reasoning

We now give in the setting of coherence the notions of p-consistency and p-entailment of Adams ([1]). Given a family of  $n$  conditional events  $\mathcal{F}_n = \{E_i|H_i, i = 1, \dots, n\}$ , we define below the notions of p-consistency and p-entailment for  $\mathcal{F}_n$ .

**Definition 2.** The family of conditional events  $\mathcal{F}_n = \{E_i|H_i, i = 1, \dots, n\}$  is *p-consistent* if and only if, for every set of lower bounds  $\{\alpha_i, i = 1, \dots, n\}$ , with  $\alpha_i \in [0, 1)$ , there exists a coherent probability assessment  $\{p_i, i = 1, \dots, n\}$  on  $\mathcal{F}_n$ , with  $p_i = P(E_i|H_i)$ , such that  $p_i \geq \alpha_i, i = 1, \dots, n$ .

**Remark 1.** Notice that p-consistency of  $\mathcal{F}_n$  can be introduced by an equivalent condition, as shown by the result below ([5, Thm 4.5],[26, Thm 8]).

**Theorem 4.** A family of conditional events  $\mathcal{F}_n$  is p-consistent if and only if the assessment  $(p_1, p_2, \dots, p_n) = (1, 1, \dots, 1)$  on  $\mathcal{F}_n$  is coherent.

**Definition 3.** A p-consistent family  $\mathcal{F}_n = \{E_i|H_i, i = 1, \dots, n\}$  *p-entails*  $B|A$ , denoted  $\mathcal{F}_n \Rightarrow_p B|A$ , if and only if there exists a nonempty subset, of  $\mathcal{F}_n$ ,  $\mathcal{S} = \{E_i|H_i, i \in J\}$  with  $J \subseteq \{1, \dots, n\}$ , such that, for every  $\alpha \in [0, 1)$ , there exists a set  $\{\alpha_i, i \in J\}$ , with  $\alpha_i \in [0, 1)$ , such that for all coherent assessments  $(z, p_i, i \in J)$  on  $\{B|A, E_i|H_i, i \in J\}$ , with  $z = P(B|A)$  and  $p_i = P(E_i|H_i)$ , if  $p_i \geq \alpha_i$  for every  $i \in J$ , then  $z \geq \alpha$ .

As we show in Theorem 6, a p-consistent family  $\mathcal{F}_n$  p-entails  $B|A$  if and only if, given any coherent assessment  $(p_1, \dots, p_n, z)$  on  $\mathcal{F}_n \cup \{B|A\}$ , from the condition  $p_1 = \dots = p_n = 1$  it follows  $z = 1$  (see also [5, Thm 4.9]). Of course, when  $\mathcal{F}_n$  p-entails  $\{B|A\}$ , there may be coherent assessments  $(p_1, \dots, p_n, z)$  with  $z \neq 1$ , but in such case  $p_i \neq 1$  for at least one index  $i$ .

We give below the notion of p-entailment between two families of conditional events  $\Gamma$  and  $\mathcal{F}$ .

**Definition 4.** Given two p-consistent finite families of conditional events  $\mathcal{F}$  and  $\Gamma$ , we say that  $\mathcal{F}$  p-entails  $\Gamma$  if  $\mathcal{F}$  p-entails  $E|H$ , for every  $E|H \in \Gamma$ .

*Transitive property:* Of course, p-entailment is transitive; that is, given three p-consistent families of conditional events  $\mathcal{F}, \Gamma, \mathcal{U}$ , if  $\mathcal{F} \Rightarrow_p \Gamma$  and  $\Gamma \Rightarrow_p \mathcal{U}$ , then  $\mathcal{F} \Rightarrow_p \mathcal{U}$ .

**Remark 2.** Notice that, from Definition 3, we trivially have that  $\mathcal{F}$  p-entails  $E|H$ , for every  $E|H \in \mathcal{F}$ ; then, by Definition 4, it immediately follows

$$\mathcal{F} \Rightarrow_p \mathcal{S}, \quad \forall \mathcal{S} \subseteq \mathcal{F}, \mathcal{S} \neq \emptyset. \quad (2)$$

### 2.3. Quasi conjunction and inclusion relation

We recall below the notion of quasi conjunction of conditional events.

**Definition 5.** Given any events  $A, H, B, K$ , with  $H \neq \emptyset, K \neq \emptyset$ , the quasi conjunction of the conditional events  $A|H$  and  $B|K$ , as defined in [1], is the conditional event  $\mathcal{C}(A|H, B|K) = (AH \vee H^c) \wedge (BK \vee K^c) | (H \vee K)$ , or equivalently  $\mathcal{C}(A|H, B|K) = (AHBK \vee AHK^c \vee H^cBK) | (H \vee K)$ . More in general, given a family of  $n$  conditional events  $\mathcal{F}_n = \{E_i|H_i, i = 1, \dots, n\}$ , the quasi conjunction of the conditional events in  $\mathcal{F}_n$  is the conditional event

$$\mathcal{C}(\mathcal{F}_n) = \mathcal{C}(E_1|H_1, \dots, E_n|H_n) = \bigwedge_{i=1}^n (E_i H_i \vee H_i^c) | \left( \bigvee_{i=1}^n H_i \right).$$

The operation of quasi conjunction is associative; that is, for every subset  $J \subset \{1, \dots, n\}$ , defining  $\Gamma = \{1, \dots, n\} \setminus J$  and

$$\mathcal{F}_J = \{E_j|H_j \in \mathcal{F}_n : j \in J\}, \quad \mathcal{F}_\Gamma = \{E_r|H_r \in \mathcal{F}_n : r \in \Gamma\},$$

it holds that  $\mathcal{C}(\mathcal{F}_n) = \mathcal{C}(\mathcal{F}_J \cup \mathcal{F}_\Gamma) = \mathcal{C}[\mathcal{C}(\mathcal{F}_J), \mathcal{C}(\mathcal{F}_\Gamma)]$ .

If  $A, H, B, K$  are logically independent, then we have ([27]):

- (i) the probability assessment  $(x, y)$  on the family  $\{A|H, B|K\}$  is coherent for every  $(x, y) \in [0, 1]^2$ ;
- (ii) given a coherent assessment  $(x, y)$  on  $\{A|H, B|K\}$ , the probability assessment  $(x, y, z)$  on  $\{A|H, B|K, \mathcal{C}(A|H, B|K)\}$ , where  $z = P[\mathcal{C}(A|H, B|K)]$ , is a coherent extension of  $(x, y)$  if and only if:  $l \leq z \leq u$ , where

$$l = \max(x + y - 1, 0); \quad u = \begin{cases} \frac{x+y-2xy}{1-xy}, & \max\{x, y\} < 1, \\ 1, & \max\{x, y\} = 1. \end{cases}$$

We observe that  $l = T_L(x, y)$  and  $u = S_0^H(x, y)$ , where  $T_L$  is the Lukasiewicz t-norm and  $S_0^H$  is the Hamacher t-conorm with parameter  $\lambda = 0$ . A general probabilistic analysis for quasi conjunction has been given in [30, 34].

Logical operations of 'conjunction', 'disjunction' and 'iterated conditioning' for conditional events have been introduced in [33].

The notion of logical inclusion among events has been generalized to conditional events by Goodman and Nguyen in [36]. We recall below this notion.

**Definition 6.** Given two conditional events  $A|H$  and  $B|K$ , we say that  $A|H$  implies  $B|K$ , denoted by  $A|H \subseteq B|K$ , if and only if  $AH$  true implies  $BK$  true and  $B^cK$  true implies  $A^cH$  true; that is

$$A|H \subseteq B|K \iff AH \subseteq BK \text{ and } B^cK \subseteq A^cH.$$

Given any conditional events  $A|H, B|K$ , we have

$$A|H = B|K \iff A|H \subseteq B|K \text{ and } B|K \subseteq A|H;$$

that is:  $A|H = B|K \iff AH = BK \text{ and } H = K$ . Moreover

$$\begin{aligned} A|H \subseteq B|K &\iff AHB^cK = H^cB^cK = AHK^c = \emptyset, \\ A|H \not\subseteq B|K &\iff AHB^cK \vee H^cB^cK \vee AHK^c \neq \emptyset. \end{aligned} \tag{3}$$

The inclusion relation among conditional events, with an extension to conditional random quantities, has been recently studied in [41].

#### 2.4. Conditional Objects

Based on a three-valued calculus of conditional objects (which are a qualitative counterpart of conditional probabilities) a logic for nonmonotonic reasoning has been proposed by Dubois and Prade in [19] (see also [3]). Such a three-valued semantics of conditional objects was first proposed for conditional events in [18]. The conditional object associated with a pair  $(p, q)$  of Boolean propositions is denoted by  $q|p$ , which reads " $q$  given  $p$ ", and concerning logical operations we can look at conditional objects as conditional events because the three-valued semantics is the same. In particular, the quasi conjunction of conditional objects exactly corresponds to quasi conjunction of conditional events and the logical entailment  $\models$  among conditional objects corresponds to the inclusion relation  $\subseteq$  among conditional events. In the non-monotonic logic developed by Dubois and Prade the quasi conjunction plays a key role, as is shown by the following definition of entailment of a conditional object from a finite conditional knowledge base  $K = \{q_i|p_i, i = 1, \dots, n\}$  (see [19], Def. 1).

**Definition 7.**  $K$  entails a conditional object  $q|p$ , denoted  $K \models q|p$ , if and only if either there exists a nonempty subset  $\mathcal{S}$  of  $K$  such that for the quasi conjunction  $\mathcal{C}(\mathcal{S})$  it holds that  $\mathcal{C}(\mathcal{S}) \models q|p$ , or  $p \models q$ .

Then, assuming  $K$  finite, the following inference rule ([1, 3, 19]) follows:

$$(QAND) \quad K \models \mathcal{C}(K).$$



### 3. Probabilistic Entailment and Quasi Conjunction

The next result, related to Adams' work, generalizes Theorem 6 given in [30] and deepens the probabilistic semantics of the QAND rule in the framework of coherence.

**Theorem 5.** Given a p-consistent family of conditional events  $\mathcal{F}_n$ , for every nonempty subfamily  $\mathcal{S} = \{E_i|H_i, i = 1, \dots, s\} \subseteq \mathcal{F}_n$ , we have

$$(a) \text{ QAND : } \mathcal{S} \Rightarrow_p \mathcal{C}(\mathcal{S}); \quad (b) \mathcal{F}_n \Rightarrow_p \mathcal{C}(\mathcal{S}). \quad (4)$$

*Proof.* (a) We preliminarily observe that the p-consistency of  $\mathcal{F}_n$  implies the p-consistency of  $\mathcal{S}$ . In order to prove that  $\mathcal{S}$  p-entails  $\mathcal{C}(\mathcal{S})$  we have to show that for every  $\varepsilon \in (0, 1]$  there exist  $\delta_1 \in (0, 1], \dots, \delta_s \in (0, 1]$  such that, for every coherent assessment  $(p_1, \dots, p_s, z)$  on  $\mathcal{S} \cup \{\mathcal{C}(\mathcal{S})\}$ , where  $p_i = P(E_i|H_i)$ ,  $z = P(\mathcal{C}(\mathcal{S}))$ , if  $p_1 \geq 1 - \delta_1, \dots, p_s \geq 1 - \delta_s$ , then  $z \geq 1 - \varepsilon$ .

We distinguish two cases: (i) the events  $E_i, H_i, i = 1, \dots, s$ , are logically independent; (ii) the events  $E_i, H_i, i = 1, \dots, s$ , are logically dependent.

(i) We recall that the case  $s = 2$ , with  $\mathcal{S} = \{E_1|H_1, E_2|H_2\}$ , has been already examined in [27], by observing that, given any coherent assessment  $(p_1, p_2, z)$  on the family  $\{E_1|H_1, E_2|H_2, \mathcal{C}(E_1|H_1, E_2|H_2)\}$  and any number  $\gamma \in [0, 1)$ , for every  $\alpha_1 \in [\gamma, 1)$ ,  $\alpha_2 \in [\gamma, 1)$ , with  $\alpha_1 + \alpha_2 \geq \gamma + 1$ , one has

$$(p_1, p_2) \in [\alpha_1, 1] \times [\alpha_2, 1] \implies z \geq \alpha_1 + \alpha_2 - 1 \geq \gamma. \quad (5)$$

We observe that, for  $\gamma = 1 - \varepsilon$ ,  $\alpha_1 = 1 - \delta_1$ ,  $\alpha_2 = 1 - \delta_2$ , with  $\alpha_1 + \alpha_2 \geq \gamma + 1$ , i.e.  $\delta_1 + \delta_2 \leq \varepsilon$ , formula (5) becomes

$$p_1 \geq 1 - \delta_1, p_2 \geq 1 - \delta_2 \implies z \geq 1 - \delta_1 + 1 - \delta_2 - 1 \geq 1 - \varepsilon.$$

More in general, denoting by  $\mathcal{L}_\gamma$  the set of the coherent assessments  $(p_1, \dots, p_s)$  on  $\mathcal{S}$  such that, for each  $(p_1, \dots, p_s) \in \mathcal{L}_\gamma$ , one has  $P[\mathcal{C}(\mathcal{S})] \geq \gamma$ , it can be proved (see [30], Theorem 4; see also [34], Theorem 9) that

$$\mathcal{L}_\gamma = \{(p_1, \dots, p_s) \in [0, 1]^s : p_1 + \dots + p_s \geq \gamma + s - 1\}.$$

In particular, given any  $\varepsilon > 0$ , it is

$$\mathcal{L}_{1-\varepsilon} = \{(p_1, \dots, p_s) \in [0, 1]^s : p_1 + \dots + p_s \geq s - \varepsilon\}.$$

Then, given any positive vector  $(\delta_1, \dots, \delta_s)$  in the set

$$\Delta_\varepsilon = \{(\delta_1, \dots, \delta_s) : \delta_1 + \dots + \delta_s \leq \varepsilon\},$$

if  $(p_1, \dots, p_s, z)$  is a coherent assessment on  $\mathcal{S} \cup \{\mathcal{C}(\mathcal{S})\}$  such that

$$p_1 \geq 1 - \delta_1, p_2 \geq 1 - \delta_2, \dots, p_s \geq 1 - \delta_s,$$

it follows  $p_1 + \dots + p_s \geq s - \varepsilon$ , so that  $(p_1, \dots, p_s) \in \mathcal{L}_{1-\varepsilon}$ , and hence  $z = P[\mathcal{C}(\mathcal{S})] \geq 1 - \varepsilon$ . Therefore  $\mathcal{S} \Rightarrow_p \mathcal{C}(\mathcal{S})$ .

(ii) We observe that, denoting by  $\Pi_s$  the set of coherent assessments on  $\mathcal{S}$ , in the case of logical independence it holds that  $\Pi_s = [0, 1]^s$ , while in case of some logical dependencies among the events  $E_i, H_i, i = 1, \dots, s$  we have  $\Pi_s \subseteq [0, 1]^s$ . Then  $\mathcal{L}_{1-\varepsilon} = \{(p_1, \dots, p_s) \in \Pi_s : p_1 + \dots + p_s \geq s - \varepsilon\}$ , with  $\mathcal{L}_{1-\varepsilon} \subseteq \{(p_1, \dots, p_s) \in [0, 1]^s : p_1 + \dots + p_s \geq s - \varepsilon\}$ , and with  $\mathcal{L}_{1-\varepsilon} \neq \emptyset$  by p-consistency of  $\mathcal{F}_s$ . Then, by the same reasoning as in case (i), we still obtain  $\mathcal{S} \Rightarrow_p \mathcal{C}(\mathcal{S})$ .

(b) By Remark 2, for each nonempty subfamily  $\mathcal{S}$  of  $\mathcal{F}_n$ , we have that  $\mathcal{F}_n$  p-entails  $\mathcal{S}$ ; then, as  $\mathcal{S}$  p-entails  $\mathcal{C}(\mathcal{S})$ , by applying Definition 3 with  $B|A = \mathcal{C}(\mathcal{S})$ , it follows  $\mathcal{F}_n \Rightarrow_p \mathcal{C}(\mathcal{S})$ .  $\square$

The next result characterizes in the setting of coherence Adams' notion of p-entailment of a conditional event  $E|H$  from a family  $\mathcal{F}_n$ ; moreover, it provides a probabilistic semantics to the notion of entailment given in Definition 7 for conditional objects. We observe that the equivalence of assertions 1 and 5 were already given (without proof) in [30, Thm 7].

**Theorem 6.** Let be given a p-consistent family  $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\}$  and a conditional event  $E|H$ . The following assertions are equivalent:

1.  $\mathcal{F}_n$  p-entails  $E|H$ ;
2. The assessment  $\mathcal{P} = (1, \dots, 1, z)$  on  $\mathcal{F} = \mathcal{F}_n \cup \{E|H\}$ , where  $P(E_i|H_i) = 1, i = 1, \dots, n, P(E|H) = z$ , is coherent if and only if  $z = 1$ ;
3. The assessment  $\mathcal{P} = (1, \dots, 1, 0)$  on  $\mathcal{F} = \mathcal{F}_n \cup \{E|H\}$ , where  $P(E_i|H_i) = 1, i = 1, \dots, n, P(E|H) = 0$ , is not coherent;
4. Either there exists a nonempty  $\mathcal{S} \subseteq \mathcal{F}_n$  such that  $\mathcal{C}(\mathcal{S})$  implies  $E|H$ , or  $H \subseteq E$ .
5. There exists a nonempty  $\mathcal{S} \subseteq \mathcal{F}_n$  such that  $\mathcal{C}(\mathcal{S})$  p-entails  $E|H$ .

*Proof.* We will prove that  $1. \Rightarrow 2. \Rightarrow 3. \Rightarrow 4. \Rightarrow 5. \Rightarrow 1.$

(1.  $\Rightarrow$  2.) Assuming that  $\mathcal{F}_n$  p-entails  $E|H$ , then  $EH \neq \emptyset$ , so that the assessment  $z = 1$  on  $E|H$  is coherent; moreover, the assessment  $(1, \dots, 1, z)$  on  $\mathcal{F}_n \cup \{E|H\}$ , where  $z = P(E|H)$ , is coherent if and only if  $z = 1$ . In fact,

if by absurd the assessment  $(1, \dots, 1, z)$  were coherent for some  $z < 1$ , then given any  $\varepsilon$ , such that  $1 - \varepsilon > z$ , the condition

$$P(E_i|H_i) = 1, \quad i = 1, \dots, n \implies P(E|H) > 1 - \varepsilon,$$

which is necessary for p-entailment of  $E|H$  from  $\mathcal{F}_n$ , would be not satisfied.

(2.  $\Rightarrow$  3.) It immediately follows by the previous step, when  $z = 0$ .

(3.  $\Rightarrow$  4.) As the assessment  $\mathcal{P} = (1, \dots, 1, 0)$  on  $\mathcal{F} = \mathcal{F}_n \cup \{E|H\}$  is not coherent, by applying Algorithm 1 to the pair  $(\mathcal{F}, \mathcal{P})$ , at a certain iteration, say the  $k$ -th one, the initial system  $\Sigma_k$  will be not solvable and the algorithm will stop. The system  $\Sigma_k$  will be associated with a pair, say  $(\mathcal{U}_k, \mathcal{P}_k)$ , where  $\mathcal{U}_k = \mathcal{S}_k \cup \{E|H\}$ , with  $\mathcal{S}_k \subseteq \mathcal{F}_n$ , and where  $\mathcal{P}_k = (1, \dots, 1, 0)$  is the (incoherent) sub-vector of  $\mathcal{P}$  associated with  $\mathcal{U}_k$ . We have two cases: (i)  $\mathcal{S}_k \neq \emptyset$ ; (ii)  $\mathcal{S}_k = \emptyset$ .

(i) For the sake of simplicity, we set  $\mathcal{S}_k = \{E_1|H_1, \dots, E_s|H_s\}$ , with  $s \leq n$ ; then, we denote by  $C_1, \dots, C_m$  the constituents generated by the family  $\mathcal{S}_k \cup \{E|H\}$  and contained in  $H_1 \vee \dots \vee H_s \vee H$ . Now, we will prove that  $\mathcal{C}(\mathcal{S}_k) \subseteq E|H$ .

We have  $\mathcal{C}(\mathcal{S}_k) = (E_1 H_1 \vee H_1^c) \wedge \dots \wedge (E_s H_s \vee H_s^c) | (H_1 \vee \dots \vee H_s)$  and, if it were  $\mathcal{C}(\mathcal{S}_k) \not\subseteq E|H$ , then equivalently, by applying formula (3) to the conditional events  $\mathcal{C}(\mathcal{S}_k), E|H$ , there would exist at least a constituent, say  $C_1$ , of the following kind:

(a)  $C_1 = B_1 A_1 \dots B_r A_r A_{r+1}^c \dots A_s^c E^c H$ ,  $1 \leq r \leq s$ , or

(b)  $C_1 = H_1^c H_2^c \dots H_s^c E^c H$ , or

(c)  $C_1 = B_1 A_1 \dots B_r A_r A_{r+1}^c \dots A_s^c H^c$ ,  $1 \leq r \leq s$ ,

where  $B_i | A_i = E_{j_i} | H_{j_i}$ ,  $i = 1, \dots, s$ , for a suitable permutation  $(j_1, \dots, j_s)$  of  $(1, \dots, s)$ .

For each one of the three cases, (a), (b), (c), the vector  $(\lambda_1, \lambda_2, \dots, \lambda_m) = (1, 0, \dots, 0)$ , associated with the constituents  $C_1, C_2, \dots, C_m$ , would be a solution of the system  $\Sigma_k$ ; then,  $\Sigma_k$  would be solvable, which would be a contradiction; hence, it cannot exist any constituent of kind (a), or (b), or (c); therefore,  $\mathcal{C}(\mathcal{S}_k) \subseteq E|H$ . Hence the assertion 4 is true for  $\mathcal{S} = \mathcal{S}_k$ .

(ii) First of all we observe that, concerning  $E$  and  $H$ , if  $EH = \emptyset$ , then the unique coherent assessment on  $E|H$  is  $P(E|H) = 0$ ; if  $EH \neq \emptyset$  and  $H \not\subseteq E$ , then the assessment  $P(E|H) = p$  on  $E|H$  is coherent for every  $p \in [0, 1]$ ; if  $H \subseteq E$ , then the unique coherent assessment on  $E|H$  is  $P(E|H) = 1$ . Now, as  $\mathcal{S}_k = \emptyset$ , the algorithm stops with  $\mathcal{U}_k = \{E|H\}$ ; then, the assessment  $P(E|H) = 0$  is incoherent, which amounts to  $H \subseteq E$ .

(4.  $\Rightarrow$  5.) If  $\mathcal{C}(\mathcal{S}) \subseteq E|H$  for some nonempty  $\mathcal{S} \subseteq \mathcal{F}_n$ , then, observing that by

p-consistency of  $\mathcal{F}_n$  the assessment  $P[\mathcal{C}(\mathcal{S})] = 1$  is coherent,  $\mathcal{C}(\mathcal{S})$  p-entails  $E|H$ . Otherwise, if  $H \subseteq E$ , then the unique coherent assessment on  $E|H$  is  $P(E|H) = 1$  and trivially  $\mathcal{C}(\mathcal{S})$  p-entails  $E|H$  for every nonempty  $\mathcal{S} \subseteq \mathcal{F}_n$ . (5.  $\Rightarrow$  1.) Assuming that  $\mathcal{C}(\mathcal{S})$  p-entails  $E|H$  for some nonempty  $\mathcal{S} \subseteq \mathcal{F}_n$ , by recalling Theorem 5, we have  $\mathcal{F}_n \Rightarrow_p \mathcal{C}(\mathcal{S}) \Rightarrow_p E|H$ . Therefore, by the transitive property of p-entailment, we have  $\mathcal{F}_n \Rightarrow_p E|H$ .  $\square$

**Remark 3.** Notice that, given two conditional events  $A|B, E|H$ , with  $AB \neq \emptyset$  (so that the family  $\{A|B\}$  is p-consistent), by applying Theorem 6 with  $n = 1$ ,  $\mathcal{F}_1 = \{A|B\}$ , it follows

$$A|B \Rightarrow_p E|H \iff A|B \subseteq E|H \text{ or } H \subseteq E.$$

We observe that p-consistency of  $\mathcal{F}_n \cup \{E|H\}$  is not sufficient for the p-entailment of  $E|H$  from  $\mathcal{F}_n$ . More precisely, we have

**Theorem 7.** Given a p-consistent family of  $n$  conditional events  $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\}$  and a conditional event  $E|H$ , the following assertions are equivalent:

- a) the family  $\mathcal{F}_n \cup \{E|H\}$  is p-consistent;
- b) exactly one of the following alternatives holds:
  - (i)  $\mathcal{F}_n$  p-entails  $E|H$ ;
  - (ii) the assessment  $\mathcal{P} = (1, \dots, 1, z)$  on  $\mathcal{F}_n \cup \{E|H\}$ , where  $P(E_i|H_i) = 1, i = 1, \dots, n, P(E|H) = z$ , is coherent for every  $z \in [0, 1]$ .

*Proof.* (a  $\Rightarrow$  b) Assuming  $\mathcal{F}_n \cup \{E|H\}$  p-consistent, if statement (i) holds, by Theorem 6 the assessment  $\mathcal{P}_0 = (1, \dots, 1, 0)$  on  $\mathcal{F} = \mathcal{F}_n \cup \{E|H\}$  is not coherent; hence, statement (ii) does not hold. If (i) doesn't hold, by Theorem 6 the assessment  $\mathcal{P}_0 = (1, \dots, 1, 0)$  on  $\mathcal{F}_n \cup \{E|H\}$  is coherent. Moreover, by p-consistency of  $\mathcal{F}_n \cup \{E|H\}$  the assessment  $\mathcal{P}_1 = (1, \dots, 1, 1)$  on  $\mathcal{F}_n \cup \{E|H\}$  is coherent. Hence, by the extension Theorem of conditional probabilities (see also [4], Theorem 7) the assessment  $\mathcal{P} = (1, \dots, 1, z)$  on  $\mathcal{F}_n \cup \{E|H\}$  is coherent for every  $z \in [0, 1]$ ; in other words, statement (ii) holds.

(b  $\Rightarrow$  a) If statement (i) holds, then by Theorem 6 the assessment  $(1, \dots, 1, 1)$  on  $\mathcal{F}_n \cup \{E|H\}$  is coherent; hence  $\mathcal{F}_n \cup \{E|H\}$  is p-consistent. If statement (ii) holds, then again the assessment  $(1, \dots, 1, 1)$  on  $\mathcal{F}_n \cup \{E|H\}$  is coherent; hence  $\mathcal{F}_n \cup \{E|H\}$  is p-consistent.  $\square$

When  $\mathcal{F}_n \cup \{E|H\}$  is not p-consistent, both statements (i) and (ii), in Theorem 7, do not hold. We observe that, given any pair  $(\mathcal{F}_n, E|H)$ , if  $\mathcal{F}_n$  is p-consistent, then we have the following three possible cases:

- (c<sub>1</sub>)  $\mathcal{F}_n \cup \{E|H\}$  is p-consistent and  $\mathcal{F}_n$  p-entails  $\{E|H\}$ ;
- (c<sub>2</sub>)  $\mathcal{F}_n \cup \{E|H\}$  is p-consistent and  $\mathcal{F}_n$  does not p-entail  $\{E|H\}$ ;
- (c<sub>3</sub>)  $\mathcal{F}_n \cup \{E|H\}$  is not p-consistent.

These three cases are characterized in the next result.

**Theorem 8.** Given a p-consistent family of  $n$  conditional events  $\mathcal{F}_n$  and a further conditional event  $E|H$ , let  $\mathcal{P} = (1, \dots, 1, z)$  be a probability assessment on  $\mathcal{F}_n \cup \{E|H\}$ , where  $P(E_i|H_i) = 1, i = 1, \dots, n, P(E|H) = z$ . Then, exactly one of the following statements is true:

- (a<sub>1</sub>)  $\mathcal{P} = (1, \dots, 1, z)$  is coherent if and only if  $z = 1$ ;
- (a<sub>2</sub>)  $\mathcal{P} = (1, \dots, 1, z)$  is coherent for every  $z \in [0, 1]$ ;
- (a<sub>3</sub>)  $\mathcal{P} = (1, \dots, 1, z)$  is coherent if and only if  $z = 0$ .

*Proof.* We show that (c<sub>1</sub>) is equivalent to (a<sub>1</sub>), (c<sub>2</sub>) is equivalent to (a<sub>2</sub>) and (c<sub>3</sub>) is equivalent to (a<sub>3</sub>).

The case (c<sub>1</sub>), by Theorem 6, amounts to say that the assessment  $\mathcal{P} = (1, \dots, 1, z)$  is coherent if and only if  $z = 1$ , which is statement (a<sub>1</sub>).

The case (c<sub>2</sub>) amounts to say that the assessment  $\mathcal{P} = (1, \dots, 1, z)$  is coherent for every  $z \in [0, 1]$ , which is statement (a<sub>2</sub>). Indeed, if  $\mathcal{F}_n \cup \{E|H\}$  is p-consistent and  $E|H$  is not p-entailed from  $\mathcal{F}_n$ , then by condition (ii) in Theorem 7 the assessment  $\mathcal{P} = (1, \dots, 1, z)$  is coherent for every  $z \in [0, 1]$ . Conversely, if the assessment  $\mathcal{P} = (1, \dots, 1, z)$  is coherent for every  $z \in [0, 1]$ , then the assessments  $\mathcal{P}_1 = (1, \dots, 1, 1)$  and  $\mathcal{P}_0 = (1, \dots, 1, 0)$  on  $\mathcal{F}_n \cup \{E|H\}$  are both coherent and hence  $\mathcal{F}_n \cup \{E|H\}$  is p-consistent and  $E|H$  is not p-entailed from  $\mathcal{F}_n$ .

The case (c<sub>3</sub>) amounts to say that the assessment  $(1, \dots, 1, 1)$  on  $\mathcal{F}_n \cup \{E|H\}$  is not coherent; that is, the assessment  $(1, \dots, 1, 0)$  on  $\mathcal{F}_n \cup \{E^c|H\}$  is not coherent; that is, by Theorem 6,  $\mathcal{F}_n$  p-entails  $E^c|H$ , or equivalently, the assessment  $(1, \dots, 1, p)$  on  $\mathcal{F}_n \cup \{E^c|H\}$  is coherent if and only if  $p = 1$ , which amounts to say that the assessment  $(1, \dots, 1, z)$  on  $\mathcal{F}_n \cup \{E|H\}$  is coherent if and only if  $z = 0$ , which is statement (a<sub>3</sub>).  $\square$

We observe that, if the assessment  $(1, \dots, 1, z)$  on  $\mathcal{F}_n \cup \{E|H\}$  is coherent for some  $z \in (0, 1)$ , then by the previous theorem the assessment  $(1, \dots, 1, z)$  is coherent for every  $z \in [0, 1]$ .

#### 4. The class $\mathcal{K}$

In this section we deepen the analysis of the quasi conjunction and the Goodman-Nguyen inclusion relation. Given a family  $\mathcal{F}_n$  and a further conditional event  $E|H$ , we set

$$\mathcal{K}(\mathcal{F}_n, E|H) = \{\mathcal{S} \subseteq \mathcal{F}_n, \mathcal{S} \neq \emptyset : \mathcal{C}(\mathcal{S}) \subseteq E|H\}. \quad (6)$$

As the family  $\mathcal{F}_n$  is finite, the class  $\mathcal{K}(\mathcal{F}_n, E|H)$  is finite too. For the sake of simplicity, we simply denote  $\mathcal{K}(\mathcal{F}_n, E|H)$  by  $\mathcal{K}$ . In what follows we will study the class  $\mathcal{K}$ , by giving some results which prove the properties listed below.

##### Properties of class $\mathcal{K}$

1. Given a nonempty subset  $\mathcal{S}$  of  $\mathcal{F}_n$  and a probability assessment  $\mathcal{P} = (1, \dots, 1, 0)$  on  $\mathcal{S} \cup \{E|H\}$ , where  $P(E_i|H_i) = 1$  for each  $E_i|H_i \in \mathcal{S}$  and  $P(E|H) = 0$ , we have:  $\mathcal{S} \in \mathcal{K} \iff \mathcal{P} \notin \mathcal{I}$ , where  $\mathcal{I}$  is the convex hull associated with the pair  $(\mathcal{S} \cup \{E|H\}, \mathcal{P})$ ;
2. it may happen that the class  $\mathcal{K}$  is empty and  $\mathcal{F}_n \Rightarrow_p E|H$ ;
3. the class  $\mathcal{K}$  is *additive*:  $\mathcal{S}' \in \mathcal{K}, \mathcal{S}'' \in \mathcal{K} \implies \mathcal{S}' \cup \mathcal{S}'' \in \mathcal{K}$ ;
4. given any  $\mathcal{S} \in \mathcal{K}, \mathcal{U} \notin \mathcal{K}$ , if  $\mathcal{S} \subset \mathcal{U}$ , then  $\mathcal{U} \setminus \mathcal{S} \notin \mathcal{K}$ ;
5. given any nonempty subsets  $\mathcal{S}$  and  $\Gamma$  of  $\mathcal{F}_n$ , if  $\mathcal{C}(\mathcal{S}) \subseteq \mathcal{C}(\Gamma)$  and  $\mathcal{S} \cup \Gamma \in \mathcal{K}$ , then  $\mathcal{S} \in \mathcal{K}$ ;
6. if  $H \not\subseteq E$  and  $\mathcal{F}_n \Rightarrow_p E|H$ , then the class  $\mathcal{K}$  is nonempty and has a greatest element  $\mathcal{S}^*$ ;
7. for every subset  $\mathcal{S} \in \mathcal{K}$ , we have:  $\mathcal{S} \Rightarrow_p E|H$ .

Property 1 follows by showing that, for any given nonempty subset  $\mathcal{S}$  of  $\mathcal{F}_n$ , the relation  $\mathcal{C}(\mathcal{S}) \subseteq E|H$ , that is  $\mathcal{S} \in \mathcal{K}$ , is equivalent to the condition  $\mathcal{P} \notin \mathcal{I}$  as proved in the following result.

**Theorem 9.** Given a p-consistent family of  $s$  conditional events  $\mathcal{S} = \{E_1|H_1, \dots, E_s|H_s\}$ , with  $s \geq 1$ , and a further conditional event  $E|H$ , let  $\mathcal{P} = (1, \dots, 1, 0)$  be a probability assessment on  $\mathcal{F} = \mathcal{S} \cup \{E|H\}$ . Moreover, let  $\mathcal{I}$  be the convex hull of the points  $Q_h$  associated with the pair  $(\mathcal{F}, \mathcal{P})$  and let  $\Sigma$  be the starting system when applying Algorithm 1. We have

$$\mathcal{P} \notin \mathcal{I}, \text{ i.e., } \Sigma \text{ unsolvable} \iff \mathcal{C}(\mathcal{S}) \subseteq E|H. \quad (7)$$

*Proof.* ( $\Rightarrow$ ) If  $\Sigma$  is unsolvable, then  $\mathcal{P} = (1, \dots, 1, 0)$  is not coherent and, by applying the part ( $3 \Rightarrow 4$ ) of Theorem 6 with  $\mathcal{F}_n = \mathcal{S}$ , Algorithm 1 stops at  $\Sigma_k = \Sigma$ ,  $\mathcal{S}_k = \mathcal{S}$ . Then, we have  $\mathcal{C}(\mathcal{S}_k) = \mathcal{C}(\mathcal{S}) \subseteq E|H$ .

( $\Leftarrow$ ) If  $\Sigma$  is solvable, then the point  $\mathcal{P}$  belongs to the convex hull  $\mathcal{I}$ ; that is,  $\mathcal{P}$  is a linear convex combination of the points  $Q_h$ . Then, as  $\mathcal{P}$  is a vertex of the unitary hypercube  $[0, 1]^{s+1}$ , which contains  $\mathcal{I}$ , the condition  $\mathcal{P} \in \mathcal{I}$  is satisfied if and only if there exists a point  $Q_h$ , say  $Q_1$ , which coincides with  $\mathcal{P}$ . Then, there exists at least a constituent  $C_1$  of the kind (a), or (b), or (c), as defined in the proof of Theorem 6, and this implies that  $\mathcal{C}(\mathcal{S}) \not\subseteq E|H$ .  $\square$

We give below an example to illustrate the previous result.

**Example 1** (*Cut Rule*). Given three logically independent events  $A, B, C$ , let us consider the family  $\mathcal{S} = \{C|AB, B|A\}$  and the further conditional event  $C|A$ . Of course,  $\mathcal{S}$  p-entails  $C|A$ . Let  $\mathcal{P} = (1, 1, 0)$  be a probability assessment on  $\mathcal{F} = \mathcal{S} \cup \{C|A\}$ . The constituents contained in  $\mathcal{H}_3 = A$  are

$$C_1 = ABC, C_2 = ABC^c, C_3 = AB^cC, C_4 = AB^cC^c, ;$$

The associated points  $Q_h$ 's are

$$Q_1 = (1, 1, 1), Q_2 = (0, 1, 0), Q_3 = (1, 0, 1), Q_4 = (1, 0, 0).$$

In Figure 1 the convex hull  $\mathcal{I}$  of the points  $Q_h$  associated with the pair  $(\mathcal{F}, \mathcal{P})$  is shown. As  $\mathcal{C}(C|AB, B|A) = BC|A \subseteq C|A$ , the system  $\Sigma$  is not solvable; that is, as shown in Figure 1, we have  $\mathcal{P} = (1, 1, 0) \notin \mathcal{I}$ .

The property 2 is proved by the following

**Proposition 1.** Given any pair  $(\mathcal{F}_n, E|H)$ , the following conditions are compatible: (i) the class  $\mathcal{K}$  is empty; (ii)  $\mathcal{F}_n \Rightarrow_p E|H$ .

*Proof.* We observe that, if  $H \subseteq E$ , then the family  $\mathcal{F}_n$  trivially p-entails  $E|H$ ; at the same time it may happen that there doesn't exist any (nonempty) subset  $\mathcal{S}$  of  $\mathcal{F}_n$  such that  $\mathcal{C}(\mathcal{S}) \subseteq E|H$ ; for instance, if  $\mathcal{F}_n = \{B|A\}$ , with  $B|A \not\subseteq E|H$  and  $H \subseteq E$ , then  $\{B|A\}$  trivially p-entails  $E|H$  and the class  $\mathcal{K}$  is empty.  $\square$

The property 3 is proved in the next result.

**Theorem 10.** Given two nonempty subsets  $\mathcal{S}'$  and  $\mathcal{S}''$  of  $\mathcal{F}_n$  and a conditional event  $E|H$ , assume that  $\mathcal{S}' \in \mathcal{K}$ ,  $\mathcal{S}'' \in \mathcal{K}$ . Then,  $\mathcal{S}' \cup \mathcal{S}'' \in \mathcal{K}$ .

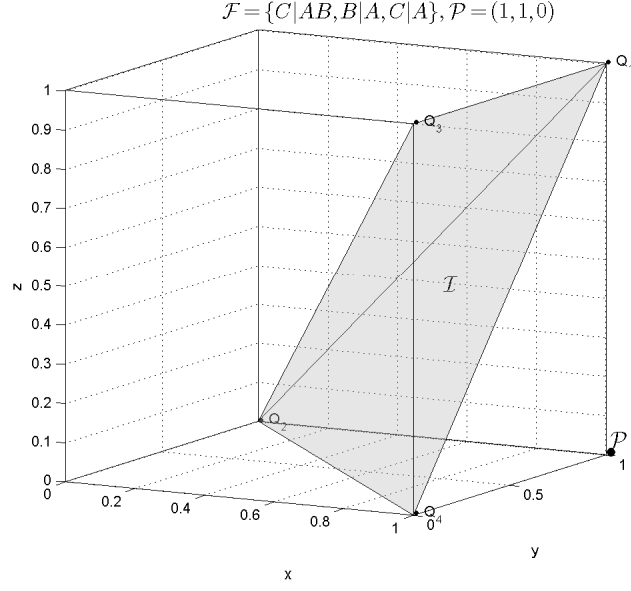


Figure 1: The convex hull  $\mathcal{I}$  associated with the pair  $(\mathcal{F}, \mathcal{P})$ .

*Proof.* By the associative property, it is  $\mathcal{C}(\mathcal{S}' \cup \mathcal{S}'') = \mathcal{C}(\mathcal{C}(\mathcal{S}'), \mathcal{C}(\mathcal{S}''))$ ; then:  
(i)  $\mathcal{C}(\mathcal{S}' \cup \mathcal{S}'')$  true implies  $\mathcal{C}(\mathcal{S}')$  true, or  $\mathcal{C}(\mathcal{S}'')$  true; hence,  $E|H$  is true;  
(ii)  $E|H$  false implies that  $\mathcal{C}(\mathcal{S}')$  and  $\mathcal{C}(\mathcal{S}'')$  are both false; hence,  $\mathcal{C}(\mathcal{S}' \cup \mathcal{S}'')$  is false. Therefore:  $\mathcal{C}(\mathcal{S}' \cup \mathcal{S}'') \subseteq E|H$ . Hence  $\mathcal{S}' \cup \mathcal{S}'' \in \mathcal{K}$ ; that is  $\mathcal{K}$  is additive.  $\square$

The property 4 is proved in the following

**Corollary 1.** Given two subsets  $\mathcal{S}$  and  $\mathcal{U}$  of  $\mathcal{F}_n$ , assume that  $\mathcal{S} \subset \mathcal{U}$ , with  $\mathcal{C}(\mathcal{S}) \subseteq E|H$ ,  $\mathcal{C}(\mathcal{U}) \not\subseteq E|H$ . Then:  $\mathcal{C}(\mathcal{U} \setminus \mathcal{S}) \not\subseteq E|H$ .

*Proof.* The proof immediately follows by Theorem 10 by observing that, if  $\mathcal{C}(\mathcal{U} \setminus \mathcal{S}) \subseteq E|H$ , then  $\mathcal{C}[\mathcal{S} \cup (\mathcal{U} \setminus \mathcal{S})] = \mathcal{C}(\mathcal{U}) \subseteq E|H$ , which contradicts the hypothesis.  $\square$

In order to prove the property 5, we recall that  $A|H \subseteq B|K$  amounts to  $H^c B^c K = A H B^c K = A H K^c = \emptyset$  (see Remark 3); thus

$$\mathcal{C}(A|H, B|K) = (A H \vee H^c B K) | (H \vee K).$$



Moreover, as shown by Table 1, it holds that

$$A|H \subseteq \mathcal{C}(A|H, B|K) \subseteq B|K. \quad (8)$$

Constituent	$A H$	$\mathcal{C}(A H, B K)$	$B K$
$H^c K^c$	Void	Void	Void
$AHBK$	True	True	True
$H^c BK$	Void	True	True
$A^c HBK$	False	False	True
$A^c HK^c$	False	False	Void
$A^c HB^c K$	False	False	False

Table 1: Truth-Table of  $A|H, \mathcal{C}(A|H, B|K), B|K$  in case  $A|H \subseteq B|K$ .

Then, we have

**Theorem 11.** Let  $\mathcal{F}_n$  be a family of  $n$  conditional events, with  $n \geq 2$ , and  $E|H$  be a further conditional event. Moreover, let  $\mathcal{S}$  and  $\Gamma$  be two nonempty subfamilies of  $\mathcal{F}_n$  such that  $\mathcal{C}(\mathcal{S}) \subseteq \mathcal{C}(\Gamma)$  and  $\mathcal{C}(\mathcal{S} \cup \Gamma) \subseteq E|H$ . Then, we have  $\mathcal{C}(\mathcal{S}) \subseteq E|H$ .

*Proof.* By the associative property of quasi conjunction we have  $\mathcal{C}(\mathcal{C}(\mathcal{S}), \mathcal{C}(\Gamma)) = \mathcal{C}(\mathcal{S} \cup \Gamma)$ ; then, by applying (8), with  $A|H = \mathcal{C}(\mathcal{S})$  and  $B|K = \mathcal{C}(\Gamma)$ , we obtain  $\mathcal{C}(\mathcal{S}) \subseteq \mathcal{C}(\mathcal{S} \cup \Gamma) \subseteq \mathcal{C}(\Gamma)$ . As  $\mathcal{C}(\mathcal{S} \cup \Gamma) \subseteq E|H$ , it follows  $\mathcal{C}(\mathcal{S}) \subseteq E|H$ .  $\square$

The property 6 is proved in the next result.

**Theorem 12.** Given a family of  $n$  conditional events  $\mathcal{F}_n$  and a further conditional event  $E|H$ , with  $H \not\subseteq E$ , assume that  $\mathcal{F}_n$  p-entails  $E|H$ . Then, the class  $\mathcal{K}$  is nonempty and has a greatest element  $\mathcal{S}^*$ .

*Proof.* Since  $\mathcal{F}_n$  p-entails  $E|H$ , by assertion 4 in Theorem 6,  $\mathcal{K}$  is nonempty; moreover, by Theorem 10,  $\mathcal{K}$  is additive. Then, denoting by  $\mathcal{S}^*$  the union of all elements of  $\mathcal{K}$ , it holds that  $\mathcal{S}^* \in \mathcal{K}$ . Of course,  $\mathcal{S}^*$  is the greatest element of  $\mathcal{K}$ ; that is,  $\mathcal{S} \subseteq \mathcal{S}^*$ , for every  $\mathcal{S} \in \mathcal{K}$ .  $\square$

The property 7 is proved in the next result.

**Proposition 2.** Given a family of  $n$  conditional events  $\mathcal{F}_n$  and a further conditional event  $E|H$ , assume that the class  $\mathcal{K}$  is nonempty. Then, for every subset  $\mathcal{S} \in \mathcal{K}$ , we have:  $\mathcal{S} \Rightarrow_p E|H$ .

*Proof.* The condition  $\mathcal{S} \in \mathcal{K}$  amounts to  $\mathcal{C}(\mathcal{S}) \subseteq E|H$ ; then, by the step (4.  $\Rightarrow$  5.) of Theorem 6, it follows that:  $\mathcal{S} \in \mathcal{K}$  implies  $\mathcal{S} \Rightarrow_p E|H$ .  $\square$

**Remark 4.** Assuming  $H \not\subseteq E$ , by Theorem 6,  $\mathcal{F}_n$  p-entails  $E|H$  if and only if there exists a nonempty subset  $\mathcal{S}_k$  of  $\mathcal{F}_n$  such that, when applying Algorithm 1 to the assessment  $\mathcal{P} = (1, \dots, 1, 0)$  on  $\mathcal{F} = \mathcal{F}_n \cup \{E|H\}$ , the system  $\Sigma_k$  associated with the family  $\mathcal{S}_k \cup \{E|H\}$  is not solvable and the algorithm will stop. In the next result we show that  $\mathcal{S}_k$  coincides with the greatest element of  $\mathcal{K}$ ,  $\mathcal{S}^*$ .

**Theorem 13.** Given a family of  $n$  conditional events  $\mathcal{F}_n$  and a further conditional event  $E|H$ , with  $H \not\subseteq E$ , assume that  $\mathcal{F}_n$  p-entails  $E|H$ . Moreover, let  $\mathcal{P} = (1, \dots, 1, 0)$  be a probability assessment on  $\mathcal{F} = \mathcal{F}_n \cup \{E|H\}$ . Then,  $\mathcal{P}$  is not coherent and by applying Algorithm 1 to the pair  $(\mathcal{F}, \mathcal{P})$ , the nonempty subset  $\mathcal{S}_k$ , associated with the iteration where Algorithm 1 stops, coincides with the greatest element  $\mathcal{S}^*$  of  $\mathcal{K}$ .

*Proof.* By assertion 3 of Theorem 6,  $\mathcal{P}$  is not coherent; moreover, by the step (3.  $\Rightarrow$  4.) of Theorem 6, we have  $\mathcal{C}(\mathcal{S}_k) \subseteq E|H$ , so that  $\mathcal{S}_k \in \mathcal{K}$  and hence  $\mathcal{S}_k \subseteq \mathcal{S}^*$ . In order to prove that  $\mathcal{S}_k = \mathcal{S}^*$ , we will show that  $\mathcal{S}_k \subset \mathcal{S}^*$  gives a contradiction. If  $\mathcal{S}_k = \mathcal{F}_n$ , then  $\mathcal{S}_k = \mathcal{S}^*$ . Assume that  $\mathcal{S}_k \subset \mathcal{F}_n$  and, by absurd, that  $\mathcal{S}_k \subset \mathcal{S}^*$ . By applying Algorithm 1 to the pair  $(\mathcal{F}, \mathcal{P})$  we obtain a partition  $\Gamma^{(1)}, \Gamma^{(2)}, \dots, \Gamma^{(k)}$ , with  $k > 1$ , such that

$$\mathcal{F}_n \cup \{E|H\} = \Gamma^{(1)} \cup \Gamma^{(2)} \cup \dots \cup \Gamma^{(k)}; \quad \Gamma^{(i)} \cap \Gamma^{(j)} = \emptyset, \text{ if } i \neq j,$$

where  $\Gamma^{(k)} = \mathcal{U}_k = \mathcal{S}_k \cup \{E|H\}$ . Then  $\mathcal{S}^* \cap \Gamma^{(k)} = \mathcal{S}_k$ . Now, by the (absurd) hypothesis  $\mathcal{S}_k \subset \mathcal{S}^*$  it would follow  $\mathcal{S}^* \cap \Gamma^{(j)} \neq \emptyset$  for at least one index  $j < k$ . Denoting by  $r$  the minimum index such that  $\mathcal{S}^* \cap \Gamma^{(r)} \neq \emptyset$  and defining

$$\mathcal{F}^{(r)} = \Gamma^{(r)} \cup \dots \cup \Gamma^{(k)}, \quad \mathcal{F}^{(r+1)} = \Gamma^{(r+1)} \cup \dots \cup \Gamma^{(k)},$$

we would have  $\mathcal{S}^* \subseteq \mathcal{F}^{(r)}$ ,  $\mathcal{S}^* \setminus \mathcal{F}^{(r+1)} \neq \emptyset$ ; moreover, the system  $\Sigma^{(r)}$  associated with the pair  $(\mathcal{F}^{(r)}, \mathcal{P}^{(r)})$  would be solvable. Defining

$$J = \{j : E_j|H_j \in \mathcal{S}^*\}, \quad \mathcal{F}_J = \mathcal{S}^* \cup \{E|H\},$$

and denoting by  $\mathcal{P}_J$  the sub-vector of  $\mathcal{P}$  associated with  $\mathcal{F}_J$ , by Theorem 2 it would follow that the system  $\Sigma_J$  associated with the pair  $(\mathcal{F}_J, \mathcal{P}_J)$  would be solvable and by Theorem 9 we would have  $\mathcal{C}(\mathcal{S}^*) \not\subseteq E|H$ , which is absurd because  $\mathcal{S}^* \in \mathcal{K}$ . Therefore  $\mathcal{S}_k = \mathcal{S}^*$ .  $\square$

Based on Theorem 4 and Theorem 13, we give below a suitably modified version of Algorithm 1, which allows to examine the following aspects:

- (i) checking for p-consistency of  $\mathcal{F}_n$ ;
- (ii) checking for p-entailment of  $E|H$  from  $\mathcal{F}_n$ ;
- (iii) computation of the greatest element  $\mathcal{S}^*$ .

**Algorithm 2.** Let be given the pair  $(\mathcal{F}_n, E|H)$ , with  $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\}$  and  $H \not\subseteq E$ .

1. Set  $\mathcal{P}_n = (1, 1, \dots, 1)$ , where  $P(E_i|H_i) = 1$ ,  $i = 1, \dots, n$ . Check the coherence of  $\mathcal{P}_n$  on  $\mathcal{F}_n$  by Algorithm 1. If  $\mathcal{P}_n$  on  $\mathcal{F}_n$  is coherent then  $\mathcal{F}_n$  is p-consistent, set  $\mathcal{F} = \mathcal{F}_n \cup \{E|H\}$ ,  $\mathcal{P} = (\mathcal{P}_n, 0)$  and go to step 2; otherwise  $\mathcal{F}_n$  is not p-consistent and procedure stops.
2. Construct the system  $\Sigma$  associated with  $(\mathcal{F}, \mathcal{P})$  and check its solvability.
3. If the system  $\Sigma$  is not solvable then  $\mathcal{F}_n$  p-entails  $E|H$ ,  $\mathcal{S}^* = \mathcal{F} \setminus \{E|H\}$  and the procedure stops; otherwise (that is,  $\Sigma$  solvable) compute the set  $I_0$  defined in formula (1).
4. If  $I_0 = \emptyset$  then  $\mathcal{F}_n$  does not p-entail  $E|H$  and the procedure stops; otherwise set  $(\mathcal{F}, \mathcal{P}) = (\mathcal{F}_0, \mathcal{P}_0)$  and go to step 2.

**Remark 5.** We point out that in Algorithm 2 the family  $\mathcal{F}_n \cup \{E|H\}$  must be intended as the family of  $n + 1$  conditional events  $\{E_1|H_1, \dots, E_n|H_n, E|H\}$  even if  $E_i|H_i = E|H$  for some  $i$ ; hence, at step 3, where  $\mathcal{F} = \mathcal{S} \cup \{E|H\}$  for some  $\mathcal{S} \subseteq \mathcal{F}_n$ , to set  $\mathcal{S}^* = \mathcal{F} \setminus \{E|H\}$  must be intended as to set  $\mathcal{S}^* = \mathcal{S}$ .

As for similar algorithms existing in literature, which analyze the problem of checking coherence and propagation, also with Algorithm 2 the checking of p-consistency and of p-entailment is intractable when the family  $\mathcal{F}_n$  is large. A detailed analysis of the different levels of complexity in this kind of problems has been given in [6]. As a further aspect, Algorithm 2 provides the greatest element (if any)  $\mathcal{S}^*$  of the class  $\mathcal{K}$ . We observe that, if in step 1  $\mathcal{F}_n$  results p-consistent, then the set  $\mathcal{S}^*$  (if any) is determined in at most  $n$  cycles of the algorithm. The example below illustrates Algorithm 2.

**Example 2.** Given four logically independent events  $A, B, C, D$ , let us consider the family  $\mathcal{F}_5 = \{C|B, B|A, A|(A \vee B), B|(A \vee B), D|A^c\}$  and the further conditional event  $C|A$ . By applying Algorithm 2 to the pair  $(\mathcal{F}_5, C|A)$ , it can be proved that the assessment  $\mathcal{P}_5 = (1, 1, 1, 1, 1)$  on  $\mathcal{F}_5$  is coherent; hence  $\mathcal{F}_5$  is p-consistent. Moreover, as

$$\mathcal{C}(\mathcal{F}_5) = (ABC \vee A^c B^c D)|\Omega \not\subseteq C|A,$$

the starting system  $\Sigma$ , associated with the pair  $(\mathcal{F}, \mathcal{P})$ , where  $\mathcal{F} = \mathcal{F}_5 \cup \{C|A\}$  and  $\mathcal{P} = (1, 1, 1, 1, 1, 0)$ , is solvable and we have  $I_0 = \{1, 2, 3, 4\}$ . The procedure goes to step 2, with  $(\mathcal{F}, \mathcal{P}) = (\mathcal{F}_0, \mathcal{P}_0)$ , where

$$\mathcal{F}_0 = \{C|B, B|A, A|(A \vee B), B|(A \vee B)\} \cup \{C|A\}, \quad \mathcal{P}_0 = (1, 1, 1, 1, 0).$$

As

$$\mathcal{C}(\{C|B, B|A, A|(A \vee B), B|(A \vee B)\}) = ABC|(A \vee B) \subset C|A,$$

the system  $\Sigma$  associated with the pair  $(\mathcal{F}_0, \mathcal{P}_0)$  is not solvable; then, by Theorem 6,  $\mathcal{F}_5$  p-entails  $C|A$  and the procedure stops, with

$$\mathcal{S}^* = \mathcal{F}_5 \setminus \{D|A^c\} = \{C|B, B|A, A|(A \vee B), B|(A \vee B)\}.$$

Moreover, by setting  $\mathcal{S}^* = \mathcal{S}_1$  and defining

$$\mathcal{S}_2 = \mathcal{S}_1 \setminus \{B|(A \vee B)\} = \{C|B, B|A, A|(A \vee B)\},$$

it holds:  $\mathcal{C}(\mathcal{S}_2) = ABC|(A \vee B) \subset B|(A \vee B)$ ; hence, by Theorem 11,  $\mathcal{C}(\mathcal{S}_2) \subset C|A$ , that is  $\mathcal{S}_2 \in \mathcal{K}$ . We also observe that, defining

$$\mathcal{S}_3 = \mathcal{S}_1 \setminus \{B|A\} = \{C|B, A|(A \vee B), B|(A \vee B)\},$$

it is:  $\mathcal{C}(\mathcal{S}_3) = ABC|(A \vee B) \subset B|A$ ; hence, by Theorem 11,  $\mathcal{C}(\mathcal{S}_3) \subset C|A$ , that is  $\mathcal{S}_3 \in \mathcal{K}$ . Finally, it could be proved that, for every nonempty subset  $\mathcal{S}$  of  $\mathcal{F}_5$ , with  $\mathcal{S} \notin \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}$ , it holds that  $\mathcal{C}(\mathcal{S}) \not\subseteq C|A$ , i.e.  $\mathcal{S} \notin \mathcal{K}$ ; therefore  $\mathcal{K} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}$  and  $\mathcal{S}_i$  p-entails  $C|A$ ,  $i = 1, 2, 3$ .

We observe that the problem of determining the class  $\mathcal{K}$  by refining the methodology applied in the previous example seems intractable because the cardinality of  $\mathcal{K}$  may be  $2^n - 1$ , as shown by the example below.

**Example 3.** Given the pair  $(\mathcal{F}_n, E|H)$ , assume that  $E_i|H_i \subseteq E|H$  for every  $i = 1, \dots, n$ . Then, it holds that: (i)  $\{E_i|H_i\} \in \mathcal{K}$  for every  $i = 1, \dots, n$ ; (ii)  $\{E_i|H_i, E_j|H_j\} \in \mathcal{K}$  for every  $\{i, j\} \subseteq \{1, \dots, n\}$ ; and so on. In this case the cardinality of  $\mathcal{K}$  is  $2^n - 1$ .

## 5. Conclusions

In this paper we have studied the probabilistic entailment in the setting of coherence. In this framework we have analyzed the role of quasi conjunction and the Goodman-Nguyen inclusion relation for conditional events. By deepening some results given in a previous paper we have shown that, given any finite family  $\mathcal{F}$  of conditional events and any nonempty subset  $\mathcal{S}$  of  $\mathcal{F}$ , the quasi conjunction  $\mathcal{C}(\mathcal{S})$  is p-entailed by  $\mathcal{F}$ . We have also examined the probabilistic semantics of QAND rule. Then, we have characterized p-entailment by many equivalent assertions. In particular, given any conditional event  $E|H$ , with  $H \not\subseteq E$ , we have shown the equivalence between p-entailment of  $E|H$  from  $\mathcal{F}$  and the existence of a nonempty subset  $\mathcal{S}$  of  $\mathcal{F}$  such that  $\mathcal{C}(\mathcal{S})$  p-entails  $E|H$ . For a pair  $(\mathcal{F}, E|H)$  we have examined some alternative theorems related with p-consistency and p-entailment. Moreover, we have introduced the (possibly empty) class  $\mathcal{K}$  of the subsets  $\mathcal{S}$  of  $\mathcal{F}$  such that  $\mathcal{C}(\mathcal{S})$  implies  $E|H$ . We have shown that the class  $\mathcal{K}$  satisfies many properties, in particular, every  $\mathcal{S} \in \mathcal{K}$  p-entails  $E|H$ ,  $\mathcal{K}$  is additive and has a greatest element which can be determined by applying Algorithm 2. Finally, we have illustrated the theoretical results and Algorithm 2, by examining an example. Interestingly, some of the results concerning the class  $\mathcal{K}$  are connected with results on a similar but different class introduced in [5, Sec 5]; further work should compare these classes and clarify the differences between them.

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